

An Approximation Algorithm for Path Computation and Function Placement in SDNs

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Abstract

We consider the task of computing (combined) function mapping and routing for requests in Software-Defined Networks (SDNs). Function mapping refers to the assignment of nodes in the substrate network to various processing stages that requests must undergo. Routing refers to the assignment of a path in the substrate network that begins in a source node of the request, traverses the nodes that are assigned functions for this request, and ends in a destination of the request.

The algorithm either rejects a request or completely serves a request, and its goal is to maximize the sum of the benefits of the served requests. The solution must abide edge and vertex capacities.

We follow the framework suggested by Even *et al.*[1] for the specification of the processing requirements and routing of requests via processing-and-routing graphs (PR-graphs). In this framework, each request has a demand, a benefit, and PR-graph.

Our main result is a randomized approximation algorithm for path computation and function placement with the following guarantee. Let m denote the number of links in the substrate network, ε denote a parameter such that $0 < \varepsilon < 1$, and opt_f denote the maximum benefit that can be attained by a fractional solution (one in which requests may be partly served and flow may be split along multiple paths). Let c_{\min} denote the minimum edge capacity, and let d_{\max} denote the maximum demand. Let Δ_{\max} denote an upper bound on the number of processing stages a request undergoes. If $c_{\min}/(\Delta_{\max} \cdot d_{\max}) = \Omega((\log m)/\varepsilon^2)$, then with probability at least $1 - \frac{1}{m} - \exp(-\Omega(\varepsilon^2 \cdot \text{opt}_f/(b_{\max} \cdot d_{\max})))$, the algorithm computes a $(1 - \varepsilon)$ -approximate solution.

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1 Introduction

Software Defined Networks (SDNs) and Network Function Virtualization (NFV) have been reinventing key issues in networking [2]. The key characteristics of these developments are: (i) separation between the data plane and the management (or control) plane, (ii) specification of the management of the network from a global view, (iii) introduction of network abstractions that provide a simple networking model, and (iv) programmability of network components.

In this paper we focus on an algorithmic problem that the network manager needs to solve in an NFV/SDN setting. This problem is called *path computation and function placing*. Path computation is simply the task of allocating paths to requests. These paths are subject to the capacity constraints of the network links and the forwarding capacity of the network nodes. In modern networks, networking is not limited to forwarding packets from sources to destinations. Requests can come in the form of flows (i.e., streams of packets from a source node to a destination node with a specified packet rate) that must undergo processing stages on their way to their destination. Examples of processing steps include: compression, encryption, firewall validation, deep packet inspection, etc. The crystal ball of NFV is the introduction of abstractions that allow one to specify, per request, requirements such as processing stages, valid locations for each processing stage, and allowable sets of links along which packets can be sent between processing stages. An important example for such goal is supporting security requirements that stipulate that unencrypted packets do not traverse untrusted links or reach untrusted nodes.

From an algorithmic point of view, the problem of path computation and function mapping combines two different optimization problems. Path computation alone (i.e., the case of pure packet forwarding without processing of packets) is an integral path packing problem. Function mapping alone (i.e., the case in which packets only need to be processed but not routed) is a load balancing problem.

To give a feeling of the problem, consider a special case of requests for streams, each of which needs to undergo the same sequence of k processing stages w_1, w_2, \dots, w_k . This means that service of a request from s_i to t_i is realized by a concatenation of $k + 1$ paths: $s_i \xrightarrow{p_0} v_1 \xrightarrow{p_1} v_2 \xrightarrow{p_2} \dots \xrightarrow{p_{k-1}} v_k \xrightarrow{p_k} t_i$, where processing stage w_i takes place in node v_i . Note that the nodes v_1, \dots, v_k need not be distinct and the concatenated path $p_0 \circ p_1 \circ \dots \circ p_k$ need not be simple. A collection of allocations that serve a set of requests not only incurs a forwarding load on the network elements, it also incurs a computational load on the nodes. The computational load is created by the need to perform the processing stages for the requests.

Previous works. Most papers on the topic resort to heuristics or non-polynomial algorithms. For example, in [6] mixed-integer programming is employed. The online version is studied in [1] in which new standby/accept service model is introduced.

Contribution and Techniques. Under reasonable assumptions (i.e., logarithmic capacity-to-demand ratio and sufficiently large optimal benefit), we present the first offline approximation algorithm for the path computation and function placing problem. Our starting point is the model of SDN requests presented in [1]. In this model, each request is represented by a special graph, called a place-and-route graph (PR-graph, in short). The PR-graph represents both the routing requirement and the processing requirements that the packets of the stream must undergo. We also build on the technique of graph products for representing valid realizations of requests [1]. We propose a fractional relaxation of the problem. The fractional relaxation consists of a set of fractional flows, each over a different product graph. Each flow is fractional in the sense that it may serve only part of a request and may split the

flow among multiple paths. We emphasize that the fractional flows do not constitute a multi-commodity flow because they are over different graphs. Nevertheless, the fractional problem is a general packing LP [4]. We solve the fractional relaxation and apply randomized rounding [4] to find an approximate solution.

Although randomized rounding is very well known and appears in many textbooks and papers, the version for the general packing problem appears only in half a page in the thesis of Raghavan [4, p. 41]. A special case with unit demands and unit benefits appears in [3]. Perhaps one of the contributions of this paper is a full description of the analysis of randomized rounding for the general packing problem.

2 Modeling Requests in SDN

In Even *et al.* [1], a model for SDN requests, based on so called place-and-route graphs (PR-graphs) and product graphs is presented. The model is quite general, and allows each request to have multiple sources and destinations, varying bandwidth demand based on processing stages, task specific capacities, prohibited locations of processing, and prohibited links for routing between processing stages, etc. We overview a simplified version of this model so that we can define the problem of path computation and function placement.

2.1 The Substrate Network

The substrate network is a fixed network of servers and communication links. The network is represented by a graph $N = (V, E)$, where V is the set of *nodes* and E is the set of *edges*. Nodes and edges have *capacities*. The capacity of an edge e is denoted by $c(e)$, and the capacity of a node $v \in V$ is denoted by $c(v)$. By scaling, we may assume that $\min_{x \in V \cup E} c(x) = 1$. We note that the network is static and undirected (namely each edge represents a bidirectional communication link), but may contain parallel edges.

2.2 Requests and PR-Graphs

Each request is specified by a tuple $r_j = (G_j, d_j, b_j, U_j)$, where the components are as follows:

1. $G_j = (X_j, Y_j)$ is a directed (acyclic) graph called the place-and-route graph (PR-graph). There is a single source (respectively, sink) that corresponds to the source (resp. destination) of the request. We denote the source and sink nodes in G_j by s_j and t_j , respectively. The other vertices correspond to services or processing stages of a request. The edges of the PR-graph are directed and indicate precedence relations between PR-vertices.
2. The demand of r_j is d_j and benefit is b_j . By scaling, we may assume that $\min_j b_j = 1$.
3. $U_j : X_j \cup Y_j \rightarrow 2^V \cup 2^E$ where $U_j(x)$ is a set of “allowed” nodes in N that can perform service x , and $U_j(y)$ is a set of “allowed” edges of N that can implement the routing requirement that corresponds to y .

2.3 The Product Network

For each request r_j , the product network $\text{pn}(N, r_j)$ is defined as follows. The node set of $\text{pn}(N, r_j)$, denoted V_j , is defined as $V_j \triangleq \cup_{y \in Y_j} (U_j(y) \times \{y\})$. We refer to the subset $U_j(y) \times \{y\}$ as the y -layer in the product graph. The edge set of $\text{pn}(N, r_j)$, denoted E_j , consists of two types of edges $E_j = E_{j,1} \cup E_{j,2}$ defined as follows.

1. *Routing edges* connect vertices in the same layer.

$$E_{j,1} = \{((u, y), (v, y)) \mid y \in Y_j, (u, v) \in U_j(y)\}.$$

2. *Processing edges* connect two copies of the same network vertex in different layers.

$$E_{j,2} = \{((v, y), (v, y')) \mid y \neq y' \text{ are 2 edges with a common endpoint } x, \text{ and } v \in U_j(x)\}.$$

2.4 Valid Realizations of SDN Requests

Consider a path \tilde{p} in the product graph $\text{pn}(N, r_j)$ that starts in the s_j -layer and ends in the t_j -layer, where s_j and t_j are the source and sink vertices of the PR-graph G_j . Such a path \tilde{p} represents the routing of request r_j from its origin to its destination and the processing stages that it undergoes. The processing edges along \tilde{p} represent nodes in which processing stages of r_j take place. The routing edges within each layer represent paths along which the request is delivered between processing stages.

► **Definition 1.** A path \tilde{p} in the product network $\text{pn}(N, r_j)$ that starts in the (source) s_j -layer and ends in the (sink) t_j -layer is a *valid realization* of request r_j .

We note that in [1] the projection of \tilde{p} to the substrate network is referred to as a valid realization. The *projection* of vertices of $\text{pn}(N, r_j)$ to vertices in N maps a vertex (u, y) to u . By the definition of the product graph, this projection maps paths in $\text{pn}(N, r_j)$ to paths in N . Consider the path p in N resulting from the projection of a path \tilde{p} in the product graph. Note that p may not be simple even if \tilde{p} is simple.

2.5 The Path Computation and Function Placement Problem (PCFP)

Notation. Consider a path \tilde{p} in the product graph $\text{pn}(N, r_j)$. The *multiplicity* of an edge $e = (u, v)$ in the substrate network N in \tilde{p} is the number of routing edges in \tilde{p} that project to e , formally:

$$\text{multiplicity}(e, \tilde{p}) \triangleq |\{y \in Y_j \mid ((u, y), (v, y)) \in \tilde{p}\}|$$

Similarly, the multiplicity of a vertex $v \in V$ in \tilde{p} is the number of processing edges in \tilde{p} that project to v , formally:

$$\text{multiplicity}(v, \tilde{p}) \triangleq |\{y \in Y_j \mid \exists y' : ((v, y), (v, y')) \in \tilde{p}\}|$$

Capacity Constraints. Let $\tilde{P} = \{\tilde{p}_i\}_{i \in I}$ denote a set of valid realizations for a subset $\{r_i\}_{i \in I} \subseteq R$ of requests. The set \tilde{P} *satisfies the capacity constraints* if

$$\begin{aligned} \sum_{i \in I} d_i \cdot \text{multiplicity}(e, \tilde{p}_i) &\leq c(e), \quad \text{for every edge } e \in E \\ \sum_{i \in I} d_i \cdot \text{multiplicity}(v, \tilde{p}_i) &\leq c(v), \quad \text{for every vertex } v \in V \end{aligned}$$

Definition of the PCFP-problem. The input in the PCFP-problem consists of a substrate network $N = (V, E)$ and a set of requests $\{r_i\}_{i \in I}$. The goal is to compute valid realizations $\tilde{P} = \{\tilde{p}_i\}_{i \in I'}$ for a subset $I' \subseteq I$ such that: (1) \tilde{P} satisfies the capacity constraints, and (2) the benefit $\sum_{i \in I'} b_i$ is maximum. We refer to the requests r_i such that $i \in I'$ as the *accepted* requests; requests r_i such that $i \in I \setminus I'$ are referred to as *rejected* requests.

3 The Approximation Algorithm for PCFP

The approximation algorithm for the PCFP-problem is described in this section. It is a variation of Raghavan's randomized rounding algorithm for general packing problems [4, Thm 4.7, p. 41] (in which the approximation ratio is $\frac{1}{e} - \sqrt{\frac{2 \ln n}{\varepsilon \cdot e \cdot \text{opt}}}$ provided that $\frac{c_{\min}}{d_{\max}} \geq \frac{\ln n}{\varepsilon}$).

3.1 Fractional Relaxation of the PCFP-problem

We now define the fractional relaxation of the PCFP-problem. Instead of assigning a valid realization \tilde{p}_i per accepted request r_i , we assign a fractional flow \tilde{f}_i in the product graph $\text{pn}(N, r_i)$. The source of flow \tilde{f}_i is the source layer (i.e., a super source that is connected to all the nodes in the source layer). Similarly, the destination of \tilde{f}_i is the destination layer. The demand of \tilde{f}_i is d_i (hence $|\tilde{f}_i| \leq d_i$). As in the integral case, the capacity constraints are accumulated across all the requests. Namely, let f_i denote the projection of \tilde{f}_i to the substrate network. The edge capacity constraint for e is $\sum_i f_i(e) \leq c(e)$. A similar constraint is defined for vertex capacities. The benefit of a fractional solution $F = \{f_i\}_i$ is $B(F) \triangleq \sum_i b_i \cdot |f_i|$.

We emphasize that this fractional relaxation is not a multi-commodity flow. The reason is that each \tilde{f}_i is over a different product graph. However, the fractional relaxation is a general packing LP.

3.2 The Algorithm

The algorithm uses a parameter $1 > \varepsilon > 0$. The algorithm proceeds as follows.

1. Divide all the capacities by $(1 + \varepsilon)$. Namely, $\tilde{c}(e) = c(e)/(1 + \varepsilon)$ and $\tilde{c}(v) = c(v)/(1 + \varepsilon)$.
2. Compute an maximum benefit fractional PCFP solution $\{\tilde{f}_i\}_i$.
3. Apply the randomized rounding procedure independently to each flow \tilde{f}_i over the product network $\text{pn}(N, r_j)$. (See Appendix B for a description of the procedure). Let p_i denote the path in $\text{pn}(N, r_i)$ (if any) that is assigned to request r_i by the randomized rounding procedure. Let $\{f'_i\}_i$ denote the projection of p_i to the substrate network. Note that each f'_i is an unsplittable all-or-nothing flow. The projection of p_i might not be a simple path in the substrate, hence the flow $f'_i(e)$ can be a multiple of the demand d_i .

3.3 Analysis of the algorithm

► **Definition 2.** The *diameter* of a PR-graph G_j is the length of a longest path in G_j from the source s_j to the destination t_j . We denote the diameter of G_j by $\Delta(G_j)$.

The diameter of G_j is well defined because G_j is acyclic for every request r_j . In all applications we are aware of, the diameter $\Delta(G_j)$ is constant (i.e., less than 5).

Notation. Let $\Delta_{\max} \triangleq \max_{j \in I} \Delta(G_j)$ denote the maximum diameter of a request. Let c_{\min} denote the minimum edge capacity, and let d_{\max} denote the maximum demand. Let opt_f denote a maximum benefit fractional PCFP solution (with respect to the original capacities $c(e)$ and $c(v)$). Let ALG denote the solution computed by the algorithm. Let $B(S)$ denote the benefit of a solutions S . Define $\beta(\varepsilon) \triangleq (1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon$.

Our goal is to prove the following theorem.¹

¹ We believe there is a typo in the analogous theorem for integral MCFs with unit demands and unit benefits in [3, Thm 11.2, p. 452] and that a factor of ε^{-2} is missing in their lower bound on the

► **Theorem 3.** Assume that $\frac{c_{\min}}{\Delta_{\max} \cdot d_{\max}} \geq \frac{4.2+\varepsilon}{\varepsilon^2} \cdot (1+\varepsilon) \cdot \ln |E|$ and $\varepsilon \in (0, 1)$. Then,

$$\Pr[\text{ALG does not satisfy the capacity constraints}] \leq \frac{1}{|E|} \quad (1)$$

$$\Pr\left[B(\text{ALG}) < \frac{1-\varepsilon}{1+\varepsilon} \cdot B(\text{opt}_f)\right] \leq e^{-\beta(-\varepsilon) \cdot B(\text{opt}_f) / (b_{\max} \cdot d_{\max})}. \quad (2)$$

We remark in asymptotic terms, the theorem states that if $\frac{c_{\min}}{\Delta_{\max} \cdot d_{\max}} = \Omega(\frac{\log |E|}{\varepsilon^2})$, then ALG satisfies the capacity constraints with probability $1 - O(1/|E|)$ and attains a benefit of $(1 - O(\varepsilon)) \cdot B(\text{opt}_f)$ with probability $1 - \exp(-\Omega(\varepsilon^2) \text{opt}_f / (b_{\max} \cdot d_{\max}))$.

Proof. The proof is based on the fact that randomized rounding is applied to each flow \tilde{f}_i independently. Thus the congestion of an edge in ALG is the sum of independent random variables. The same holds for the $B(\text{ALG})$. The proof proceeds by applying Chernoff bounds.

Proof of Eq. 1. For the sake of simplicity we assume that there are no vertex capacities (i.e., $c(v) = \infty$). The proof is based on the Chernoff bound in Theorem 5. To apply the bound, fix a substrate edge $e \in E$. Recall that $f'_i(e)$ is a flow path that is obtained by a projection of a path in the product network $\text{pn}(N, r_i)$. Let

$$X_i \triangleq \frac{f'_i(e)}{\Delta_{\max} \cdot d_{\max}}$$

$$\mu_i \triangleq \frac{\tilde{c}(e)}{\Delta_{\max} \cdot d_{\max}} \cdot \frac{\tilde{f}_i(e)}{\sum_{j \in I} \tilde{f}_j(e)}.$$

The conditions of Theorem 5 are satisfied for the following reasons. Note that $0 \leq X_i \leq 1$ because $f'_i(e) \in \{0, d_i, \dots, \Delta_{\max} \cdot d_i\}$. Also, by Claim 1, $\mathbf{E}[X_i] = \tilde{f}_i(e) / (\Delta_{\max} \cdot d_{\max})$. Since $\sum_{j \in I} \tilde{f}_j(e) \leq \tilde{c}(e)$, it follows that $\mathbf{E}[X_i] \leq \mu_i$. Finally, $\mu \triangleq \sum_{i \in I} \mu_i = \tilde{c}(e) / (\Delta_{\max} \cdot d_{\max})$.

Let $\text{ALG}(e)$ denote the load incurred on the edge e by ALG. Namely $\text{ALG}(e) \triangleq \sum_{i \in I} f'_i(e)$. Note that $\text{ALG}(e) \geq (1 + \varepsilon) \cdot \tilde{c}(e)$ iff

$$\sum_{i \in I} X_i \geq (1 + \varepsilon) \cdot \frac{\tilde{c}(e)}{\Delta_{\max} \cdot d_{\max}} = (1 + \varepsilon) \cdot \mu.$$

From Theorem 5 we conclude that:

$$\Pr[\text{ALG}(e) \geq (1 + \varepsilon) \cdot \tilde{c}(e)] \leq e^{-\beta(\varepsilon) \cdot \tilde{c}(e) / (\Delta_{\max} \cdot d_{\max})}$$

By scaling of capacities, we have $c(e) = (1 + \varepsilon) \cdot \tilde{c}(e)$. By Fact 4, $\beta(\varepsilon) \geq \frac{2\varepsilon^2}{4.2+\varepsilon}$. By the assumption $\frac{\tilde{c}(e)}{\Delta_{\max} d_{\max}} \geq \frac{4.2+\varepsilon}{\varepsilon^2} \cdot \ln |E|$. We conclude that

$$\Pr[\text{ALG}(e) \geq c(e)] \leq \frac{1}{|E|^2}.$$

Eq. 1 follows by applying a union bound over all the edges.

Proof of Eq. 2. The proof is based on the Chernoff bound stated in Theorem 6. To apply the bound, let

$$X_i \triangleq \frac{b_i \cdot |f'_i|}{b_{\max} \cdot d_{\max}}$$

$$\mu_i \triangleq \frac{b_i \cdot |\tilde{f}_i|}{b_{\max} \cdot d_{\max}}.$$

capacities.

The conditions of Theorem 6 are satisfied for the following reasons. Since $b_i \leq b_{\max}$ and $|f'_i| \leq d_{\max}$, it follows that $0 \leq X_i \leq 1$. Note that $\sum_i X_i = B(\text{ALG})/(b_{\max} \cdot d_{\max})$. By Corollary 1, $\mathbf{E}[X_i] = \mu_i$. Finally, by linearity, $\sum_i b_i \cdot |\tilde{f}_i| = \text{opt}_f/(1 + \varepsilon)$ and $\sum_i \mu_i = \frac{B(\text{opt}_f)}{(1 + \varepsilon)b_{\max} \cdot d_{\max}}$, and the theorem holds. \blacktriangleleft

3.4 Unit Benefits

We note that in the case of identical benefits (i.e., all the benefits equal one and hence $b_{\max} = 1$) one can strengthen the statement. If $B(\text{opt}_f) > c_{\min}$, then the large capacities assumption implies that $B(\text{opt}_f)/(d_{\max} \cdot b_{\max}) \geq c_{\min}/d_{\max} \geq \varepsilon^{-2} \cdot \ln |E|$. This implies that $B(\text{ALG}) \geq (1 - O(\varepsilon)) \cdot B(\text{opt}_f)$ with probability at least $1 - 1/\text{poly}(|E|)$. By adding the probabilities of the two possible failures (i.e., violation of capacities and small benefit) and taking into account the prescaling of capacities, we obtain that with probability at least $1 - O(1/\text{poly}(|E|))$, randomized rounding returns an all-or-nothing unsplittable multi-commodity flow whose benefit is at least $1 - O(\varepsilon)$ times the optimal benefit.

4 Discussion

Theorem 3 provides an upper bounds of the probability that ALG is not feasible and that $B(\text{ALG})$ is far from $B(\text{opt}_f)$. These bounds imply that our algorithm can be viewed as version of an asymptotic PTAS in the following sense. Suppose that the parameters b_{\max} and d_{\max} are not a function of $|E|$. As the benefit of the optimal solution opt_f increases, the probability that $B(\text{ALG}) \geq (1 - O(\varepsilon)) \cdot B(\text{opt}_f)$ increases. On the other hand, we need the capacity-to-demand ratio to be logarithmic, namely, $c_{\min} \geq \Omega((\Delta_{\max} \cdot d_{\max} \cdot \ln |E|)/\varepsilon^2)$. We believe that the capacity-to-demand ratio is indeed large in realistic networks.

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A Multi-Commodity Flows

Consider a directed graph $G = (V, E)$. Assume that edges have non-negative capacities $c(e)$. For a vertex $u \in V$, let $\text{out}(u)$ denote the outward neighbors, namely the set $\{y \in V \mid (u, y) \in E\}$. Similarly, $\text{in}(u) \triangleq \{x \in V \mid (x, u) \in E\}$. Consider two vertices s and t in V (called the *source* and *destination* vertices, respectively). A *flow* from s to t is a function $f : E \rightarrow \mathbb{R}^{\geq 0}$ that satisfies the following conditions:

- (i) Capacity constraints: for every edge $(u, v) \in E$, $0 \leq f(u, v) \leq c(u, v)$.
- (ii) Flow conservation: for every vertex $u \in V \setminus \{s, t\}$

$$\sum_{x \in \text{in}(u)} f(x, u) = \sum_{y \in \text{out}(u)} f(u, y).$$

The *amount* of flow delivered by the flow f is defined by

$$|f| \triangleq \sum_{y \in \text{out}(s)} f(s, y) - \sum_{x \in \text{in}(s)} f(x, s).$$

Consider a set ordered pairs of vertices $\{(s_i, t_i)\}_{i \in I}$. An element $i \in I$ is called a *commodity* as it denotes a request to deliver flow from s_i to t_i . Let $F \triangleq \{f_i\}_{i \in I}$ denote a set of flows, where each flow f_i is a flow from the source vertex s_i to the destination vertex t_i . We abuse notation, and let F denote the sum of the flows, namely $F(e) \triangleq \sum_{i \in I} f_i(e)$, for every edge e . Such a sequence is a *multi-commodity flow* if, in addition it satisfies *cumulative capacity constraints* defined by:

$$\text{for every edge } (u, v) \in E: \quad F(u, v) \leq c(u, v).$$

Demands are used to limit the amount of flow per commodity. Formally, let $\{d_i\}_{i \in I}$ denote a sequence of positive real numbers. We say that d_i is the *demand* of flow f_i if we impose the constraint that $|f_i| \leq d_i$. Namely, one can deliver at most d_i amount of flow for commodity i .

The *maximum benefit optimization problem* associated with multi-commodity flow is formulated as follows. The input consists of a (directed) graph $G = (V, E)$, edge capacities $c(e)$, a sequence source-destination pairs for commodities $\{(s_i, t_i)\}_{i \in I}$. Each commodity has a nonnegative demand d_i and benefit b_i . The goal is to find a multi-commodity flow that maximizes the objective $\sum_{(u,v) \in E} b_i \cdot |f_i|$. We often refer to this objective as the *benefit* of the multi-commodity flow. When the demands are identical and the benefits are identical, the maximum benefit problem reduces to a maximum *throughput* problem.

A multi-commodity flow is *all-or-nothing* if $|f_i| \in \{0, d_i\}$, for every commodity $i \in I$. A multi-commodity flow is *unsplittable* if the support of each flow is a simple path. (The *support* of a flow f_i is the set of edges (u, v) such that $f_i(u, v) > 0$.) We often emphasize the fact that a multi-commodity flow is not all-or-nothing or not unsplittable by saying that it is *fractional*.

B Randomized Rounding Procedure

In this section we overview the randomized rounding procedure. The presentation is based on [3]. Given an instance $F = \{f_i\}_{i \in I}$ of a fractional multi-commodity flow with demands and benefits, we are interested in finding an all-or-nothing unsplittable multi-commodity flow $F' = \{f'_i\}_{i \in I}$ such that the benefit of F' is as close to the benefit of F as possible.

► **Observation 1.** As flows along cycles are easy to eliminate, we assume that the support of every flow $f_i \in F$ is acyclic.

We employ a randomized procedure, called *randomized rounding*, to obtain F' from F . We emphasize that all the random variables used in the procedure are independent. The procedure is divided into two parts. First, we flip random independent coins to decide which commodities are supplied. Next, we perform a random walk along the support of the supplied commodities. Each such walk is a simple path along which the supplied commodity is delivered. We describe the two parts in detail below.

Deciding which commodities are supplied. For each commodity, we first decide if $|f'_i| = d_i$ or $|f'_i| = 0$. This decision is made by tossing a biased coin $bit_i \in \{0, 1\}$ such that

$$\Pr[bit_i = 1] \triangleq \frac{|f_i|}{d_i}.$$

If $bit_i = 1$, then we decide that $|f'_i| = d_i$ (i.e., commodity i is fully supplied). Otherwise, if $bit_i = 0$, then we decide that $|f'_i| = 0$ (i.e., commodity i is not supplied at all).

Assigning paths to the supplied commodities. For each commodity i that we decided to fully supply (i.e., $bit_i = 1$), we assign a simple path P_i from its source s_i to its destination t_i by following a random walk along the support of f_i . At each node, the random walk proceeds by rolling a dice. The probabilities of the sides of the dice are proportional to the flow amounts. A detailed description of the computation of the path P_i is given in Algorithm 1.

Algorithm 1 Algorithm for assigning a path P_i to flow f_i .

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1:  $P_i \leftarrow \{s_i\}$ .
2:  $u \leftarrow s_i$ 
3: while  $u \neq t_i$  do                                     ▷ did not reach  $t_i$  yet
4:    $v \leftarrow \text{choose-next-vertex}(u)$ .
5:   Append  $v$  to  $P_i$ 
6:    $u \leftarrow v$ 
7: end while
8: return  $(P_i)$ .
9: procedure  $\text{choose-next-vertex}(u, f_i)$                    ▷ Assume that  $u$  is in the support of  $f_i$ 
10:   Define a dice  $C(u, f_i)$  with  $|\text{out}(u)|$  sides. The side corresponding to an edge  $(u, v)$ 
      has probability  $f_i(u, v) / (\sum_{(u, v') \in \text{out}(u)} f_i(u, v'))$ .
11:   Let  $v$  denote the outcome of a random roll of the dice  $C(u, f_i)$ .
12:   return  $(v)$ 
13: end procedure

```

Definition of F' . Each flow $f'_i \in F'$ is defined as follows. If $bit_i = 0$, then f'_i is identically zero. If $bit_i = 1$, then f'_i is defined by

$$f'_i(u, v) \triangleq \begin{cases} d_i & \text{if } (u, v) \in P_i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, F' is an all-or-nothing unsplittable flow, as required.

C Analysis of Randomized Rounding

The presentation in this section is based on [3].

C.1 Expected flow per edge

► **Claim 1.** For every commodity i and every edge $(u, v) \in E$:

$$\Pr[(u, v) \in P_i] = \frac{f_i(u, v)}{d_i},$$

$$\mathbf{E}[f'_i(u, v)] = f_i(u, v).$$

Proof. Since

$$\mathbf{E}[f'_i(u, v)] = d_i \cdot \Pr[(u, v) \in P_i],$$

it suffices to prove the first part.

An edge (u, v) can belong to the path P_i only if $f_i(u, v) > 0$. We now focus on edges in the support of f_i . By Observation 1, the support is acyclic, hence we can sort the support in topological ordering. The claim is proved by induction on the position of an edge in this topological ordering.

The induction basis, for edges $(s_i, y) \in \text{out}(s_i)$, is proved as follows. Since the support of f_i is acyclic, it follows that $f_i(x, s_i) = 0$ for every $(x, s_i) \in \text{in}(s_i)$. Hence $|f_i| = \sum_{y \in \text{out}(s_i, f_i)} f_i(s_i, y)$. Hence,

$$\begin{aligned} \Pr[(s_i, y) \in P_i] &= \Pr[\text{bit}_i = 1] \cdot \Pr[\text{dice } C(s_i, f_i) \text{ selects } (s_i, y) \mid \text{bit}_i = 1] \\ &= \frac{|f_i|}{d_i} \cdot \frac{f_i(s_i, y)}{\sum_{y \in \text{out}(s_i, f_i)} f_i(s_i, y)} \\ &= \frac{f_i(s_i, y)}{d_i}, \end{aligned}$$

and the induction basis follows.

The induction step, for an edge (u, v) in the support of f_i such that $u \neq s_i$, is proved as follows. Vertex u is in P_i if and only if P_i contains an edge whose head is u . We apply the induction hypothesis to these incoming edges, and use flow conservation to obtain

$$\begin{aligned} \Pr[u \in P_i] &= \Pr\left[\bigcup_{x \in \text{in}(u)} (x, u) \in P_i\right] \\ &= \frac{1}{d_i} \cdot \sum_{x \in \text{in}(u)} f_i(x, u) \\ &= \frac{1}{d_i} \cdot \left(\sum_{y \in \text{out}(u)} f_i(u, y)\right). \end{aligned}$$

Now,

$$\begin{aligned} \Pr[(u, v) \in P_i] &= \Pr[u \in P_i] \cdot \Pr[\text{dice } C(u, f_i) \text{ selects } (u, v) \mid u \in P_i] \\ &= \frac{1}{d_i} \cdot \left(\sum_{y \in \text{out}(u)} f_i(u, y)\right) \cdot \frac{f_i(u, v)}{\sum_{y \in \text{out}(u)} f_i(u, y)} \\ &= \frac{f_i(u, v)}{d_i}, \end{aligned}$$

and the claim follows. ◀

By linearity of expectation, we obtain the following corollary.

► **Corollary 1.** $\mathbf{E}[|f'_i|] = |f_i|$.

D

 Mathematical Background

In this section we present material from Raghavan [5] and Young [7] about the Chernoff bounds used in the analysis of randomized rounding.

► **Fact 1.** $e^x \geq 1 + x$ and $x \geq \ln(1 + x)$ for $x > -1$.

► **Fact 2.** $(1 + \alpha)^x \leq 1 + \alpha \cdot x$, for $0 \leq x \leq 1$ and $\alpha \geq -1$.

► **Fact 3 (Markov Inequality).** For a non-negative random variable X and $\alpha > 0$, $\Pr[X \geq \alpha] \leq \frac{\mathbf{E}[X]}{\alpha}$.

► **Definition 4.** The function $\beta : (-1, \infty) \rightarrow \mathbb{R}$ is defined by $\beta(\varepsilon) \triangleq (1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon$.

► **Fact 4.** For ε such that $-1 < \varepsilon < 1$ we have $\beta(-\varepsilon) \geq \frac{\varepsilon^2}{2} \geq \beta(\varepsilon) \geq \frac{2\varepsilon^2}{4.2 + \varepsilon}$. Hence, $\beta(-\varepsilon) = \Omega(\varepsilon^2)$ and $\beta(\varepsilon) = \Theta(\varepsilon^2)$.

► **Theorem 5 (Chernoff Bound).** Let $\{X_i\}_i$ denote a sequence of independent random variables attaining values in $[0, 1]$. Assume that $\mathbf{E}[X_i] \leq \mu_i$. Let $X \triangleq \sum_i X_i$ and $\mu \triangleq \sum_i \mu_i$. Then, for $\varepsilon > 0$,

$$\Pr[X \geq (1 + \varepsilon) \cdot \mu] \leq e^{-\beta(\varepsilon) \cdot \mu}.$$

Proof. Let A denote the event that $X \geq (1 + \varepsilon) \cdot \mu$. Let $f(x) \triangleq (1 + \varepsilon)^x$. Let B denote the event that

$$\frac{f(X)}{f((1 + \varepsilon) \cdot \mu)} \geq 1.$$

Because $f(x) > 0$ and $f(x)$ is monotone increasing, it follows that $\Pr[A] = \Pr[B]$. By Markov's Inequality,

$$\Pr[B] \leq \frac{\mathbf{E}[f(X)]}{f((1 + \varepsilon) \cdot \mu)}.$$

Since $X = \sum_i X_i$ is the sum of independent random variables,

$$\begin{aligned} \mathbf{E}[f(X)] &= \prod_i \mathbf{E}[(1 + \varepsilon)^{X_i}] \\ &\leq \prod_i \mathbf{E}[1 + \varepsilon \cdot X_i] && \text{(by Fact 2)} \\ &\leq \prod_i (1 + \varepsilon \cdot \mu_i) \\ &\leq \prod_i e^{\varepsilon \cdot \mu_i} && \text{(by Fact 1)} \\ &= e^{\varepsilon \cdot \mu} \end{aligned}$$

We conclude that

$$\begin{aligned} \Pr[A] &\leq \frac{e^{\varepsilon \cdot \mu}}{f((1 + \varepsilon) \cdot \mu)} \\ &= e^{-\beta(\varepsilon) \cdot \mu}, \end{aligned}$$

and the theorem follows. ◀

We prove an analogue theorem for bounding the probability of the event that X is much smaller than μ .

► **Theorem 6** (Chernoff Bound). *Under the same premises as in Theorem 5 except that $\mathbf{E}[X_i] \geq \mu_i$, it holds that, for $1 > \varepsilon \geq 0$,*

$$\Pr[X \leq (1 - \varepsilon) \cdot \mu] \leq e^{-\beta(-\varepsilon) \cdot \mu}.$$

Proof. We repeat the proof of Theorem 5 with the required modifications. Let A denote the event that $X \leq (1 - \varepsilon) \cdot \mu$. Let $g(x) \triangleq (1 - \varepsilon)^x$. Let B denote the event that

$$\frac{g(X)}{g((1 - \varepsilon) \cdot \mu)} \geq 1.$$

Because $g(x) > 0$ and $g(x)$ is monotone decreasing, it follows that $\Pr[A] = \Pr[B]$. By Markov's Inequality,

$$\Pr[B] \leq \frac{\mathbf{E}[g(X)]}{g((1 - \varepsilon) \cdot \mu)}.$$

Since $X = \sum_i X_i$ is the sum of independent random variables,

$$\begin{aligned} \mathbf{E}[g(X)] &= \prod_i \mathbf{E}[(1 - \varepsilon)^{X_i}] \\ &\leq \prod_i \mathbf{E}[1 - \varepsilon \cdot X_i] && \text{(by Fact 2)} \\ &\leq \prod_i (1 - \varepsilon \cdot \mu_i) \\ &\leq \prod_i e^{-\varepsilon \cdot \mu_i} && \text{(by Fact 1)} \\ &= e^{-\varepsilon \cdot \mu} \end{aligned}$$

We conclude that

$$\begin{aligned} \Pr[A] &\leq \frac{e^{-\varepsilon \cdot \mu}}{g((1 - \varepsilon) \cdot \mu)} \\ &= e^{-\beta(-\varepsilon) \cdot \mu}, \end{aligned}$$

and the theorem follows. ◀